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A subsurface eccentric crack in a functionally graded coating layer on the layered half-space under an anti-plane shear impact load

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Abstract

This paper considers a subsurface crack in a functionally graded coating layer on the layered half-space subjected to an anti-plane impact load. The over-coated layer is assumed as a functionally graded material. The second kind Fredholm integral equation is obtained using Laplace transform and Fourier transform. The dynamic stress intensity factors can be obtained through the use of numerical inversion of Laplace transform technique. Numerical results are given for different values of the non-homogeneous parameter, geometric constants and material properties.

Keywords: Subsurface eccentric crack; Functionally graded material; Anti-plane shear impact; Dynamic stress intensity factor

1. Introduction

In recent years, the mechanical and thermal barrier over-coatings have been used to protect the substrate metallic part. The concept of the functionally graded materials (FGMs) has been introduced and applied to the development of aerospace structures, aircraft engines, gas turbines and military armors etc. Erdogan (1995) discussed the problem of crack growth in FGM due to fatigue, creep and stress corrosion cracking, and fracture instability.

For the non-homogeneous interfacial problems, Erdogan (1985) examined the singular behavior of the crack-tip stress field for a crack perpendicular to and intersecting the interface of two bonded non-homogeneous half-spaces under anti-plane shear loading. Delale and Erdogan (1988) solved some crack problems for a non-homogeneous interfacial layer between two dissimilar half-planes subjected to a static loading. Itou and Shima (1999) presented solutions of the axisymmetric problem for a cylindrical crack in a non-homogeneous interfacial zone between a circular elastic cylinder and an infinite elastic medium under

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mode I loading. Yang and Shih (1994) studied the non-homogeneous interlayer with a semi-infinite crack between two dissimilar orthotropic half-spaces. Noda and Jin (1993) studied the crack problems in non-homogeneous materials under thermal loading. Choi (1996) obtained SIFs by Cauchy-type singular integral equation method for a layered medium containing a crack perpendicular to the functionally graded non-homogeneous interlayer. Gu et al. (1999) presented the SIFs for the non-homogeneous material using a finite element method by the domain integral methodology. Shbeeb and Binienda (1999) solved the interface crack problem for the sandwiched FGM strip between two homogeneous dissimilar layers.

For elastodynamic response of a composite materials, Li and Tai (1991) solved the problem for a center interface crack of two isotropic dissimilar strips between isotropic dissimilar half-spaces under an anti-plane shear impact load. Sih and Chen (1981) presented some solutions for dynamic crack problems of a composite materials. For a dynamic crack problem, almost authors assumed that the interfacial layer is homogeneous layer with constant material properties. But, Babaei and Lukasiewicz (1998) assumed that the interfacial layer is a FGM layer with properties exponentially varied and they obtained the dynamic SIF (DSIF) for a crack in a FGM layer between isotropic half-spaces under an anti-plane shear impact load.

Lee and Erdogan (1995) showed the solutions of the thermal SIF for the interface crack between the FGM over-coating and the substrate under a steady-state heat flow. Cai and Bao (1998) analyzed, using a finite element method, the crack bridging in the FGM coating which is a ceramic/metal composite with its gradation characterized by local volume fractions of metal and ceramic phases. Jin and Batra (1996) solved the interface crack problems of four coating models such as single layered homogeneous coating, double layered piece-wise homogeneous coating, single layered FGM coating and double layered over-coating with FGM bottom coat, under an anti-plane shear loading. However, these previous solutions for layered over-coating crack models consider only the static loading cases.

In this paper, we will consider a subsurface crack in a non-homogeneous over-coating on the layered half-space subjected to an anti-plane impact load. The over-coated layer is assumed FGM. Solutions are expressed in terms of the Fredholm integral equation using Laplace transform and Fourier transform. DSIF can be obtained by numerical inversion of Laplace transform technique of Miller and Guy (1966). Numerical results are given for different values of the non-homogeneous parameter, geometric constants and material properties.

2. Governing equation

A subsurface crack in the over-coated FGM layer on the layered half-space as shown in Fig. 1. The crack length is $2a$, and the over-coated layer height is H_1. For the formulation, we consider the over-coated layer as two bonded layers with an interface crack, namely layers 1 and 2. The variation of the shear modulus, μ, and the density, ρ, in the over-coated layer are assumed by:

$$\begin{aligned} \mu_1 &= \mu_0 \exp[\beta(h_1 + y)], \\ \mu_2 &= \mu_0 \exp[\beta(h_1 + y)], \\ \rho_1 &= \rho_0 \exp[\beta(h_1 + y)], \\ \rho_2 &= \rho_0 \exp[\beta(h_1 + y)], \end{aligned} \tag{1}$$

where μ_0 and ρ_0 are constants and β is a non-homogeneous parameter expressed in the form,

$$\beta = \frac{1}{h_1 + h_2} \ln\left(\frac{\mu_3}{\mu_0}\right). \tag{2}$$

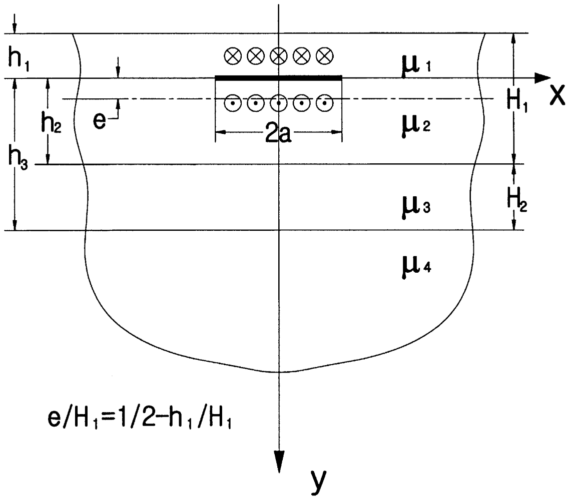

Fig. 1. Crack model of the layered half-space with the over-coated non-homogeneous layer.

Under the anti-plane shear impact load, the displacement should be in the form,

$$u_x^{(i)} = u_y^{(i)} = 0, \qquad u_z^{(i)} = w_i(x, y, t). \tag{3}$$

The equation of motion can be expressed in the form,

$$\nabla^2 w_i + \beta_i \frac{\partial w_i}{\partial y} = \frac{1}{C_{2i}^2} \frac{\partial^2 w_i}{\partial t^2}, \tag{4}$$

where

$$\begin{aligned} C_{2i} &= C_0 = \sqrt{\mu_0/\rho_0}, & \beta_i &= \beta \quad (i = 1, 2), \\ C_{2i} &= \sqrt{\mu_i/\rho_i}, & \beta_i &= 0 \quad (i = 3, 4) \end{aligned} \tag{5}$$

and C_{2i} is a shear wave velocity in the ith layer.

We consider the following boundary conditions,

$$\tau_{yz}^{(1)}(x, 0, t) = \tau_{yz}^{(2)}(x, 0, t) = -\tau_0 H(t), \quad |x| \leqslant a, \tag{6}$$

$$\left. \begin{aligned} & w_1(x, 0, t) - w_2(x, 0, t) = 0, \\ & \tau_{yz}^{(1)}(x, 0, t) - \tau_{yz}^{(2)}(x, 0, t) = 0, \end{aligned} \right. \quad |x| \geqslant a, \tag{7}$$

where τ_0 is constant and $H(t)$ denotes the Heaviside unit step function. The material interfaces are assumed to be perfectly bonded. The shear stress and displacement should be continuous across the interfaces in the forms

$$\tau_{yz}^{(i)}(x, h_i, t) = \tau_{yz}^{(i+1)}(x, h_i, t) \quad (i = 2, 3), \tag{8}$$

$$w_i(x, h_i, t) = w_{i+1}(x, h_i, t) \quad (i = 2, 3). \tag{9}$$

The upper surface ($y = -h_1$) is traction free,

$$\tau_{yz}^{(1)}(x, -h_1, t) = 0. \tag{10}$$

3. Analysis

Define a Laplace transform pair in the forms,

$$\Phi^*(p) = \int_0^\infty \Phi(t)\exp(-pt)\,\mathrm{d}t, \tag{11}$$

$$\Phi(t) = \frac{1}{2\pi i}\int_{\mathrm{Br}} \Phi^*(p)\exp(pt)\,\mathrm{d}p, \tag{12}$$

where the integral in Eq. (12) is taken over the Bromwich path. The Laplace transform Eq. (4) yields:

$$\nabla^2 w_i^* + \beta_i \frac{\partial w_i^*}{\partial y} = \frac{p^2}{C_{2i}^2} w_i^*. \tag{13}$$

The deflection $w_i^*(x,y,p)$ in Laplace transform domain depends only on the space variables x and y. If the Fourier cosine transform is employed on the variable x, Eq. (13) reduces to an ordinary differential equation, with solutions of the followings,

$$w_i^*(x,y,p) = \frac{2}{\pi}\int_0^\infty \{A^{(i)}(s,p)\mathrm{e}^{(\Omega_{1i}y)} + B^{(i)}(s,p)\mathrm{e}^{(\Omega_{2i}y)}\}\cos(sx)\,\mathrm{d}s \quad (i = 1,2), \tag{14a}$$

$$w_3^*(x,y,p) = \frac{2}{\pi}\int_0^\infty \{A^{(3)}(s,p)\mathrm{e}^{(-\gamma_3 y)} + B^{(3)}(s,p)\mathrm{e}^{(\gamma_3 y)}\}\cos(sx)\,\mathrm{d}s, \tag{14b}$$

$$w_4^*(x,y,p) = \frac{2}{\pi}\int_0^\infty A^{(4)}(s,p)\mathrm{e}^{(-\gamma_4 y)}\cos(sx)\,\mathrm{d}s, \tag{14c}$$

where

$$\begin{aligned}\Omega_{1i} &= -\left(\frac{\beta_i}{2}\right) - \sqrt{\left(\frac{\beta_i}{2}\right)^2 + s^2 + \left(\frac{p}{C_0}\right)^2},\\ \Omega_{2i} &= -\left(\frac{\beta_i}{2}\right) + \sqrt{\left(\frac{\beta_i}{2}\right)^2 + s^2 + \left(\frac{p}{C_0}\right)^2},\end{aligned} \tag{15}$$

$$\gamma_j = \sqrt{s^2 + \left(\frac{p}{C_{2j}}\right)^2} \qquad (j = 3,4). \tag{16a}$$

The corresponding anti-plane shear stress distributions in Laplace transform become in the forms,

$$\tau_{yz}^{*(i)}(x,y,p) = \frac{2}{\pi}\int_0^\infty \mu_i\{\Omega_{1i}A^{(i)}(s,p)\mathrm{e}^{(\Omega_{1i}y)} + \Omega_{2i}B^{(i)}(s,p)\mathrm{e}^{(\Omega_{2i}y)}\}\cos(sx)\,\mathrm{d}s \quad (i = 1,2), \tag{16b}$$

$$\tau_{yz}^{*(3)}(x,y,p) = -\frac{2}{\pi}\int_0^\infty \mu_3\gamma_3\{A^{(3)}(s,p)\mathrm{e}^{(-\gamma_3 y)} - B^{(3)}(s,p)\mathrm{e}^{(\gamma_3 y)}\}\cos(sx)\,\mathrm{d}s, \tag{16c}$$

$$\tau_{yz}^{*(4)}(x,y,p) = -\frac{2}{\pi}\int_0^\infty \mu_4\gamma_4 A^{(4)}(s,p)\mathrm{e}^{(-\gamma_4 y)}\cos(sx)\,\mathrm{d}s. \tag{16d}$$

Boundary conditions, Eqs. (6)–(10), become the following forms in the Laplace transform domain,

$$\tau_{yz}^{*(1)}(x,0,p) = \tau_{yz}^{*(2)}(x,0,p) = -\frac{\tau_0}{p}, \quad |x| \leqslant a, \tag{17}$$

$$\begin{aligned} &w_1^*(x,0,p) - w_2^*(x,0,p) = 0, \\ &\tau_{yz}^{*(1)}(x,0,p) - \tau_{yz}^{*(2)}(x,0,p) = 0, \end{aligned} \quad |x| \geqslant a, \tag{18}$$

$$\tau_{yz}^{*(i)}(x,h_i,p) = \tau_{yz}^{*(i+1)}(x,h_i,p) \quad (i = 2,3), \tag{19}$$

$$w_i^*(x,h_i,p) = w_{i+1}^*(x,h_i,p) \quad (i = 2,3), \tag{20}$$

$$\tau_{yz}^{*(1)}(x,-h_1,p) = 0. \tag{21}$$

From Eqs. (14a) to (18), we obtain a pair of dual integral equations in the forms,

$$\int_0^\infty A(s,p)\cos(sx)\,\mathrm{d}s = 0, \quad |x| \geqslant a, \tag{22}$$

$$\int_0^\infty sf_3(s,p)A(s,p)\cos(sx)\,\mathrm{d}s = -\frac{\pi\tau_0}{p\mu_0\exp(\beta h_1)}, \quad |x| < a, \tag{23}$$

where

$$f_3(s,p) \equiv \frac{2}{s}\left(\Omega_{12}\frac{A^{(2)}(s,p)}{A(s,p)} + \Omega_{22}\frac{B^{(2)}(s,p)}{A(s,p)}\right), \tag{24}$$

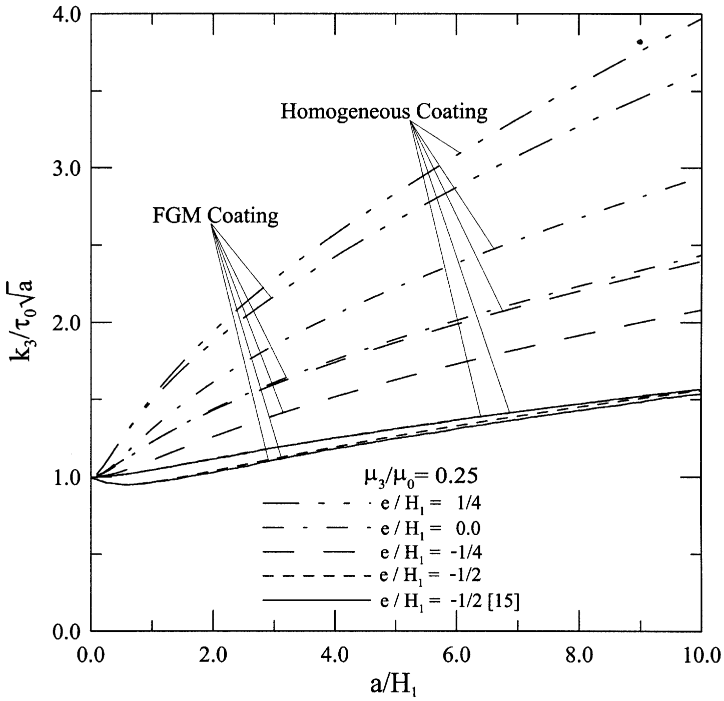

Fig. 2. The normalized SIFs vs. the crack length for various crack eccentricities in case of two layers.

$A^{(2)}(s,p)$ and $B^{(2)}(s,p)$ are given by the matrix equation in the form,

$$\begin{pmatrix} A^{(2)}(s,p) \\ B^{(2)}(s,p) \\ A^{(3)}(s,p) \\ B^{(3)}(s,p) \\ A^{(4)}(s,p) \end{pmatrix} = A(s,p)[M]^{-1} \begin{pmatrix} \exp(-\Omega_{21}h_1) - \exp(-\Omega_{11}h_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{25}$$

M matrix is shown in Appendix.

The unknown function $A(s,p)$ can be defined by Copson's method (Copson, 1961):

$$A(s,p) = -\frac{\pi^2 a^2 \tau_0}{2\mu_0 p \exp(\beta h_1)} \int_0^\infty \sqrt{\xi}\Psi^*(\xi,p) J_0(sa\xi)\, d\xi, \tag{26}$$

where $A(s,p)$ satisfies Eqs. (22)–(24). We obtained the following second kind Fredholm integral equation,

$$\Psi^*(\xi,p) + \int_0^1 \Psi^*(\eta,p) K(\xi,\eta,p)\, d\eta = \sqrt{\xi}, \tag{27}$$

where

$$K(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^\infty \alpha \left[f_3\left(\frac{\alpha}{a},p\right) - 1 \right] J_0(\alpha\eta) J_0(\alpha\xi)\, d\alpha, \tag{28}$$

$J_0(\ )$ is the zero order Bessel function of the first kind. Since the function $A(s,p)$ is known Eqs. (26)–(28), the entire stress field can be obtained.

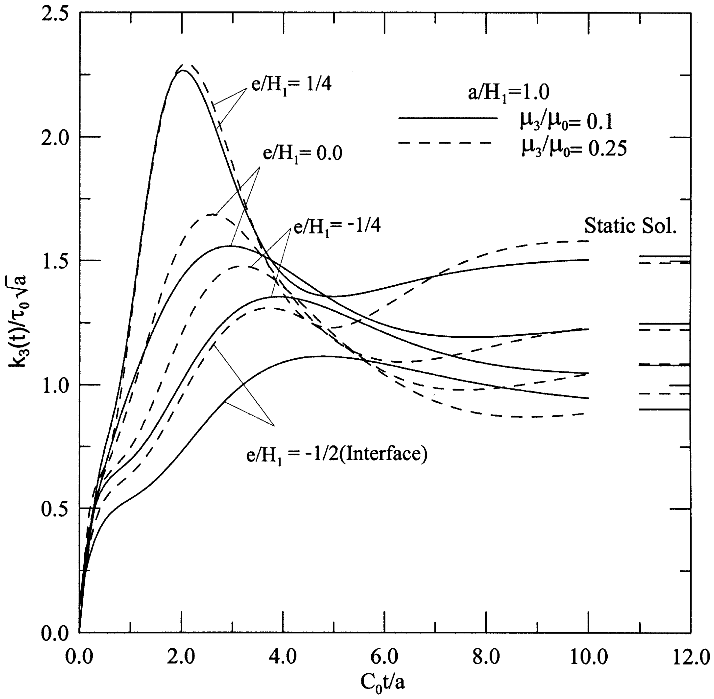

Fig. 3. The normalized DSIFs vs. time for various crack eccentricities in the case of two layers.

In the anti-plane shear impact load case, DSIF, $k_3(t)$, near the crack tip is presented in the form,

$$k_3(t) = \frac{\tau_0\sqrt{a}}{2\pi i}\int_{\mathrm{Br}} \frac{\Psi^*(1,p)}{p} \mathrm{e}^{pt}\,\mathrm{d}p. \tag{29}$$

4. Numerical results and discussion

The Fredholm integral equation in Laplace transform domain is solved numerically using Gauss Lagueere and Gauss Legendre technique. The inversion of Laplace transform is accomplished by the numerical procedure developed by Miller and Guy (1966). The solutions at large $T^*(=C_0t/a)$ converge to the static solutions. The static SIF, K_3, is obtained by applying Tauberian's final value theorem (Sneddon, 1972) in the form,

$$\lim_{T^*\to\infty} k_3(T^*) \simeq K_3 = \Psi_s(1)\tau_0\sqrt{a}, \tag{30}$$

where

$$\Psi_s(\xi) = \lim_{p\to 0} \Psi^*(\xi,p). \tag{31}$$

4.1. Case 1: Two layer case of $\mu_3 = \mu_4$ and $\rho_3 = \rho_4$

Fig. 2 shows $k_3/\tau_0\sqrt{a}$ vs. a/H_1 for various crack eccentricity ($e/H_1 \equiv 1/2 - h_1/H_1$) in cases of the homogeneous over-coating and the FGM over-coating. The SIFs give the unit value approximately at very

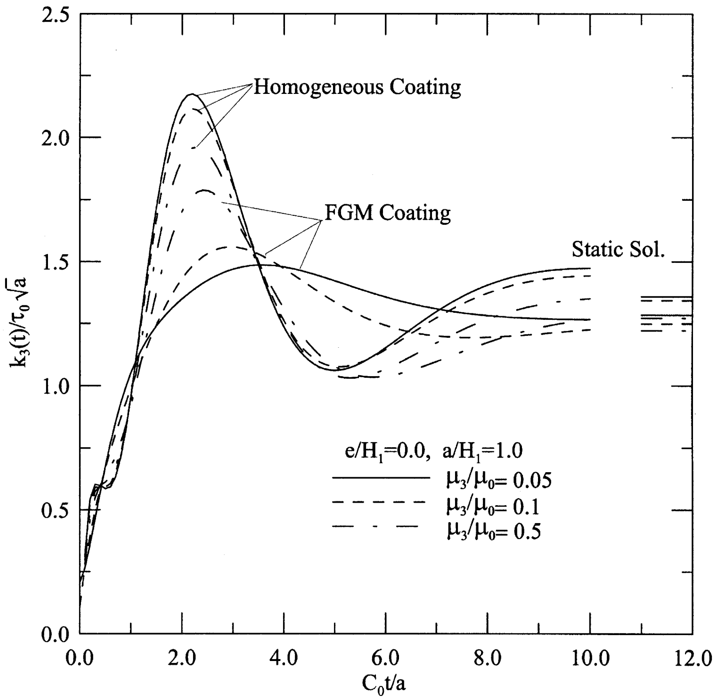

Fig. 4. The normalized DSIFs vs. time for various shear moduli in case of two layers.

small crack lengths ($a/H_1 \ll 1$). For the interface crack, $e/H_1 = -1/2$, our results are in good agreement with those of Jin and Batra (1996).

Fig. 3 displays $k_3(t)/\tau_0\sqrt{a}$ vs. C_0t/a in the FGM over-coating model in cases of various shear moduli and crack eccentricities. The DSIFs decrease with the decrease of the eccentricity, because of the variation of the shear modulus with crack location. When $\mu_3/\mu_0 < 1$, the larger μ_3/μ_0 values give the larger DSIFs.

Fig. 4 shows the DSIFs vs. the normalized time in the cases of the homogeneous over-coating and the FGM over-coating, respectively, and the ratios of shear modulus, $\mu_3/\mu_0 = 0.05, 0.1, 0.5$, respectively, for the cases of $e/H_1 = 0.0$ and $a/H_1 = 1.0$. The peak values of the normalized DSIFs for the homogeneous over-coating and larger than those for the FGM over-coating. The phenomenon is natural because the wave velocity in the homogeneous coating layer is higher than in the FGM layer as the shear modulus in the homogeneous coating layer is larger than in the FGM coating layer. The dynamic inertia effects for the FGM coating models decrease with the decrease of the ratio of shear modulus, in case of $\mu_3/\mu_0 < 1$, but the trend for the homogeneous coating is opposite. For the homogeneous coating model, it is noted that small oscillations of the DSIF curves for the lower C_0t/a values are observed and those small peaks are generated by the arrival of reflected waves from the free surface boundaries to the crack tip region (Chen and Sih, 1977).

4.2. Case 2: Three layer case of $\rho_3 = \rho_4$

Fig. 5 displays $k_3/\tau_0\sqrt{a}$, vs. H_2/H_1, in cases of various crack eccentricities and shear moduli. The values of the SIFs increase with the increase of H_2/H_1, and then converge to constant values. Fig. 6 shows $k_3/\tau_0\sqrt{a}$ vs. a/H_1. with various crack eccentricities. For an interface crack, $e/H_1 = -1/2$, our results of $H_2/H_1 = 100.0$ are in agreement with those of Jin and Batra (1996) in case of $H_2/H_1 \to \infty$.

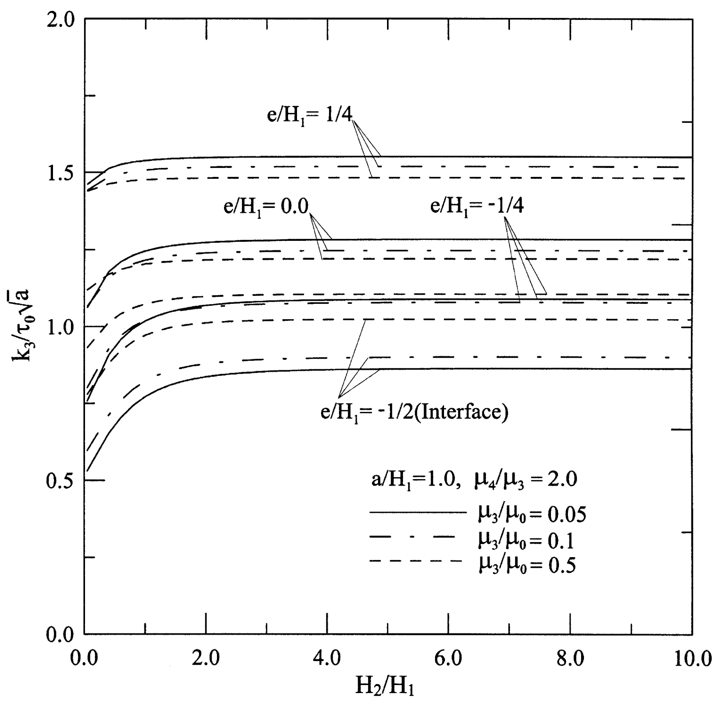

Fig. 5. The normalized SIFs vs. internal layer thickness ratio in case of three layers.

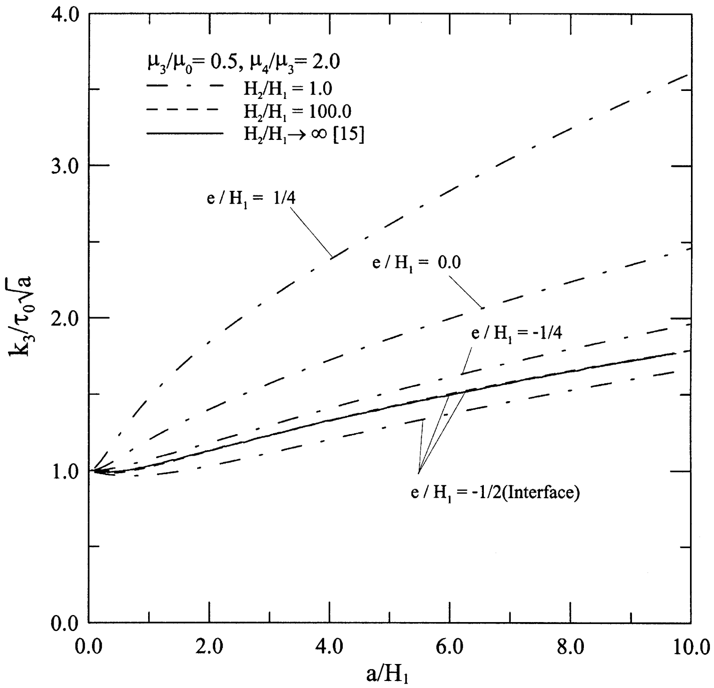

Fig. 6. The normalized SIFs vs. crack length in case of three layers.

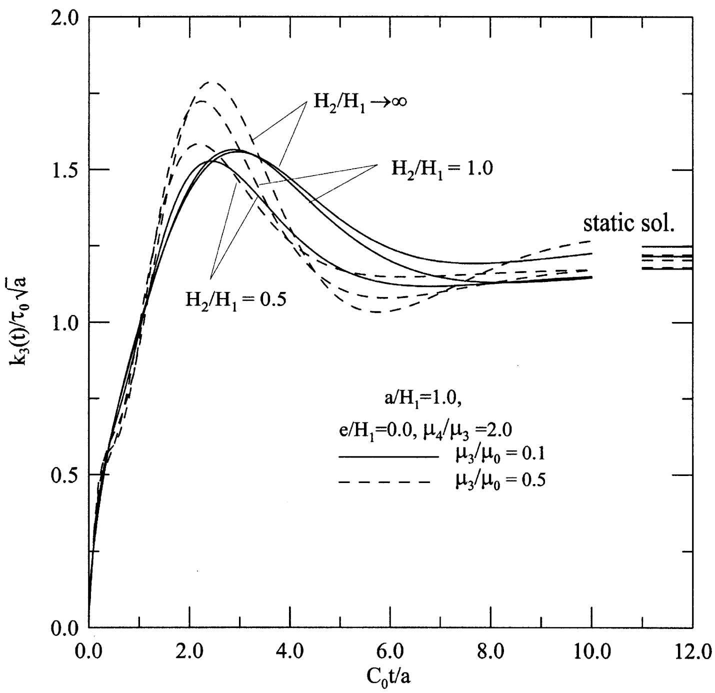

Fig. 7. The normalized DSIFs vs. the internal layer thickness ratio in case of three layers.

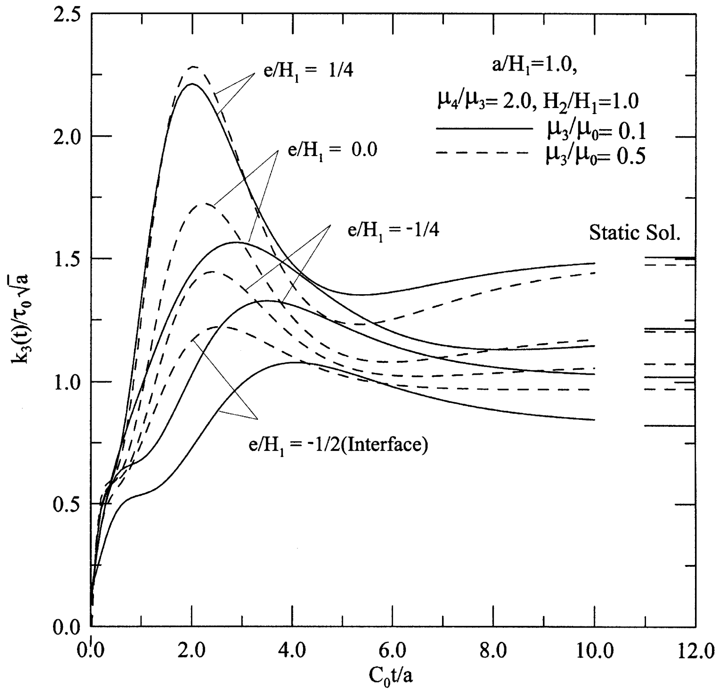

Fig. 8. The normalized DSIFs vs. the crack eccentricity in the case of three layers.

Fig. 7 displays $k_3(t)/\tau_0\sqrt{a}$ vs. C_0t/a in cases of $H_2/H_1 = 0.5$, 1.0 and ∞. The peak values of DSIFs increase with the increase of H_2/H_1 in case of $\mu_3/\mu_0 = 0.5$. But they do not show the difference in case of $\mu_3/\mu_0 = 0.1$.

Fig. 8 shows $k_3(t)/\tau_0\sqrt{a}$ vs. C_0t/a for various crack eccentricities. The trend is same an in case of two layers.

5. Conclusion

The elastodynamic problem of a subsurface crack in the over-coated non-homogeneous layer on the layered isotropic half-space under an anti-plane impact load is analyzed by integral transform technique. Fredholm integral equation is solved numerically, and the inversion of Laplace transform is accomplished by the method of Miller and Guy. The results are expressed in terms of the DSIFs. For the interface crack, our static results are in good agreement with the previous solutions. The values of DSIFs decrease with the decrease of the crack eccentricity. The dynamic inertia effects for the homogeneous over-coating are more affected than for the FGM over-coating. The DSIFs increase with the increase of the internal layer thickness. The effect of the layer thickness ratio on DSIFs decrease with the decrease of shear modulus ratio of the internal isotropic layer to the top surface of FGM.

Appendix A

$$[M] = \begin{bmatrix} M_a(1) & 0 & 0 \\ M_b(2) & M_c(2) & 0 \\ 0 & M_d(3) & M_o(3) \end{bmatrix}, \tag{A.1}$$

where

$$M_a = \left[\left(1-\frac{\Omega_{11}}{\Omega_{21}}\right)\exp(-\Omega_{11}h_1), \left(\frac{\Omega_{21}}{\Omega_{11}}-1\right)\exp(-\Omega_{21}h_1)\right], \tag{A.2}$$

$$M_b(2) = \begin{bmatrix} \exp(\Omega_{12}h_2) & \exp(\Omega_{22}h_2) \\ \Omega_{12}\exp[\beta(h_1+h_2)+\Omega_{12}h_2] & \Omega_{22}\exp[\beta(h_1+h_2)+\Omega_{22}h_2] \end{bmatrix}, \tag{A.3}$$

$$M_c(2) = \begin{bmatrix} -\exp(-\gamma_3 h_2) & -\exp(\gamma_3 h_2) \\ \frac{\mu_3}{\mu_0}\gamma_3\exp(-\gamma_3 h_2) & -\frac{\mu_3}{\mu_0}\gamma_3\exp(\gamma_3 h_2) \end{bmatrix}, \tag{A.4}$$

$$M_d(3) = \begin{bmatrix} \exp(-\gamma_3 h_3) & \exp(\gamma_3 h_3) \\ -\gamma_3\exp(-\gamma_3 h_3) & \gamma_3\exp(\gamma_3 h_3) \end{bmatrix}, \tag{A.5}$$

$$M_o(3) = \begin{bmatrix} -\exp(-\gamma_4 h_3) \\ \frac{\mu_3}{\mu_0}\gamma_4\exp(-\gamma_4 h_3) \end{bmatrix}. \tag{A.6}$$

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